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ON AUTOMORPHISM GROUPS OF C^*-ALGEBRAS

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I. **Introduction.** Let X be a compact Hausdorff space and $\mathscr{L}(\boldsymbol{H})$ the algebra of all bounded operators on a Hilbert space $\boldsymbol{H}$. $C(X; \mathscr{L}(\boldsymbol{H}))$ is the C^*-algebra of all continuous $\mathscr{L}(\boldsymbol{H})$-valued functions defined on X, with the sup-norm. The study of automorphisms of such algebras has been initiated by Kadison and Ringrose in [13, IV, Example d] where the case $\boldsymbol{H}$ finite dimensional is studied in some detail. A similar study of the case $\boldsymbol{H}$ infinite dimensional (and X a separable compact Hausdorff space) has been made by E. C. Lance [18] and independently by moi-même.

In the present work we study center-fixing automorphisms of $\mathfrak{A}=C(X; \boldsymbol{B})$, where X is an arbitrary compact Hausdorff space, and $\boldsymbol{B}$ a C^*-algebra. We introduce the notion of "locally-inner" automorphisms of $C(X; \boldsymbol{B})$, by "localizing" the definition of inner automorphisms. The precise definition requires several preliminary properties of center-fixing automorphisms and is given in §III.

The locally-inner automorphisms form a subgroup of Aut $(\mathfrak{A})$, the automorphism group of $\mathfrak{A}$, which we denote by loc-Inn $(\mathfrak{A})$. By the definition of loc-Inn $(\mathfrak{A})$ we will have an inclusion loc-Inn $(\mathfrak{A}) \subseteq$ CF $(\mathfrak{A})$, the center-fixing automorphism group of $\mathfrak{A}$. If $\boldsymbol{H}$ is finite dimensional, then loc-Inn $(C(X; \mathscr{L}(\boldsymbol{H}))=$ CF $(C(X; \mathscr{L}(\boldsymbol{H})))$ [13]. For a general Hilbert space $\boldsymbol{H}$ we establish (Theorem 3.5) an isomorphism of topological groups

$$\text{loc-Inn}\,(C(X; \mathscr{L}(\boldsymbol{H}))) = C(X; \text{Aut}\,(\mathscr{L}(\boldsymbol{H})))$$

where $C(X; \text{Aut}\,(\mathscr{L}(\boldsymbol{H})))$ denotes the group of continuous maps $f\colon X \to \text{Aut}\,(\mathscr{L}(\boldsymbol{H}))$. This occupies the major portion of §III.

With the aid of a theorem of Kallman [14], we find (Corollary 3.18) that CF $(\mathfrak{A})=$ loc-Inn $(\mathfrak{A})$ when X is a *separable* compact Hausdorff space and $\mathfrak{A}= C(X; \mathscr{L}(\boldsymbol{H}))$. Every inner automorphism is locally-inner and Inn $(\mathfrak{A})$, the inner automorphism group of $\mathfrak{A}$, is a normal subgroup of loc-Inn $(\mathfrak{A})$. As a consequence of the results of §III (Theorem 4.1) and a result of Kuiper [16] we obtain for infinite-dimensional $\boldsymbol{H}$ a natural isomorphism of groups

$$\text{loc-Inn}\,(\mathfrak{A})/\text{Inn}\,(\mathfrak{A}) = \check{H}^2(X; \boldsymbol{Z}),$$

where $\check{H}^2(X; \boldsymbol{Z})$ denotes the 2nd Čech cohomology group of X with coefficients in

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the ring of integers $\boldsymbol{Z}$. When X is separable, combining the above results with those of Lance [18] we obtain (Corollary 4.2)

$$\mathrm{CF}(\mathfrak{A}) = \pi(\mathfrak{A}) = \text{loc-Inn}(\mathfrak{A})$$

as a consequence. We also obtain for separable X the equality

$$\mathrm{Inn}(\mathfrak{A}) = \mathrm{Aut}_0(\mathfrak{A}) \quad (= \text{the identity component of } \mathrm{Aut}(\mathfrak{A})).$$

These results overlap with those of Lance in [18].

In §V we discuss relations between the carefully ideal preserving automorphisms and CF ($\mathfrak{A}$) where $\mathfrak{A}$ is $C(X; \boldsymbol{B})$. Specifically, when X is separable we obtain

$$\tau_0(\mathfrak{A}) = \pi(\mathfrak{A}) = \mathrm{CF}(\mathfrak{A})$$

where $\tau_0(\mathfrak{A})$ denotes the group of carefully ideal preserving automorphisms of $\mathfrak{A}$ (compare the results of Lance in [17]).

The presentation below is an outgrowth of the author's Thesis submitted to Yale University in partial fullfillment of the requirements of the Ph.D. degree.

II. **Preliminaries.** In this paper it is assumed that the reader is familiar with the basic results of the theory of C^*-algebras and their *-representations as may be found in [20], early chapters of [2], [3] or [4]. Also those propositions for which no proofs are given are elementary and may be verified by the reader.

Definition and Conventions. Throughout this paper a C^*-algebra will always mean a C^*-algebra over the complex numbers C, with identity which is usually denoted by I.

If $\mathfrak{A}$ is a C^*-algebra an automorphism of $\mathfrak{A}$ is an isomorphism of complex vector spaces $\alpha: \mathfrak{A} \to \mathfrak{A}$ such that

(1) $\alpha(A^*) = \alpha(A)^*$ for all $A \in \mathfrak{A}$,

(2) $\alpha(AB) = \alpha(A)\alpha(B)$ for all $A, B \in \mathfrak{A}$,

(3) $\alpha(I) = I$.

The set of all automorphisms of $\mathfrak{A}$ is denoted by Aut ($\mathfrak{A}$); it is a group in a natural way, the group operation being a composition of mappings.

If S is a complex Banach space then $L(S)$ denotes the algebra of all bounded linear operators on S. If $T \in L(S)$ we define

$$\|T\| = \sup_{s \in S; \|s\| \leq 1} \|T(s)\|.$$

This defines a norm on $\mathscr{L}(S)$.

If $\mathfrak{A}$ is a C^*-algebra then $\mathfrak{A}$ is semisimple with unique topology (in the sense of [19]) and thus any automorphism of $\mathfrak{A}$ is continuous in the norm topology of $\mathfrak{A}$. Hence Aut $(\mathfrak{A}) \subset \mathscr{L}(\mathfrak{A})$ and we may equip Aut ($\mathfrak{A}$) with the relative topology. In this topology Aut ($\mathfrak{A}$) becomes a topological group.

Note that any automorphism of $\mathfrak{A}$ is an isometry [6].

Definition of various subgroups of Aut $(\mathfrak{A})$. Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\boldsymbol{H}$. An automorphism $\alpha \in$ Aut $(\mathfrak{A})$ is said to be *extendible* if there is an automorphism $\bar{\alpha}$ of the weak operator closure $\mathfrak{A}^-$ of $\mathfrak{A}$ such that $\bar{\alpha}|_{\mathfrak{A}} = \alpha: \mathfrak{A} \to \mathfrak{A}$. An automorphism $\alpha \in$ Aut $(\mathfrak{A})$ is said to be *spatial* if there is a unitary operator U on $\boldsymbol{H}$ such that for all $A \in \mathfrak{A}$, $\alpha(A) = UAU^*$. $\alpha \in$ Aut $(\mathfrak{A})$ is said to be *weakly-inner* if it is spatial and U can be chosen in the weak operator closure of $\mathfrak{A}$.

An automorphism $\alpha \in$ Aut $(\mathfrak{A})$ is *inner* if there exists a unitary element $U \in \mathfrak{A}$ such that $\alpha(A) = UAU^*$ for all $A \in \mathfrak{A}$. Note that an inner automorphism of $\mathfrak{A}$ is weakly-inner in any faithful $*$-representation of $\mathfrak{A}$.

Let $\mathfrak{A}$ be an abstract C^*-algebra. If φ is a faithful $*$-representation of $\mathfrak{A}$ on a Hilbert space, φ-Ext $(\mathfrak{A})$ denotes the group of those elements α of Aut $(\mathfrak{A})$ for which $\varphi\alpha\varphi^{-1}$ is extendible; $\sigma_\varphi(\mathfrak{A})$ denotes the group of those elements α of Aut $(\mathfrak{A})$ for which $\varphi\alpha\varphi^{-1}$ is spatial; φ-Inn $(\mathfrak{A})$ denotes the group of those elements α of Aut $(\mathfrak{A})$ for which $\varphi\alpha\varphi^{-1}$ is weakly-inner. Let $\pi(\mathfrak{A}) = \bigcap_\varphi \varphi\text{-Inn}(\mathfrak{A})$, where the intersection is taken over all faithful $*$-representations φ of $\mathfrak{A}$. $\pi(\mathfrak{A})$ is called *the permanently weakly-inner* (*or π-inner*) automorphisms of $\mathfrak{A}$. The group of all inner automorphisms of $\mathfrak{A}$ is denoted by Inn $(\mathfrak{A})$. The connected component of $I \in$ Aut $(\mathfrak{A})$ (in the uniform topology) is denoted by $\text{Aut}_0(\mathfrak{A})$.

These subgroups of Aut $(\mathfrak{A})$ were defined and studied by Kadison and Ringrose in [13].

An automorphism $\alpha \in$ Aut $(\mathfrak{A})$ is said to be *center-fixing* if α leaves the elements of the center $Z(\mathfrak{A})$ elementwise fixed; i.e. $\alpha(A) = A$ for all $A \in Z(\mathfrak{A})$.

PROPOSITION. *Let $\mathfrak{A}$ be a C^*-algebra. Then the set of all center-fixing automorphisms of $\mathfrak{A}$ forms a subgroup of* Aut $(\mathfrak{A})$. □

The subgroup of Aut $(\mathfrak{A})$ consisting of all the center-fixing automorphisms of $\mathfrak{A}$ is denoted by CF $(\mathfrak{A})$, and is referred to as the center-fixing automorphism group of $\mathfrak{A}$. Note that Inn $(\mathfrak{A}) \subseteq$ CF $(\mathfrak{A})$.

Some results from the work of Kadison and Ringrose [13]. It is convenient to have available some of the results of [13] concerning inclusion relations between the subgroups of Aut $(\mathfrak{A})$ introduced above. The remainder of this section is devoted to a summary of the relevant facts from [13].

THEOREM A [13, THEOREM 7]. *Let $\mathfrak{A}$ be a C^*-algebra and $\alpha \in$* Aut $(\mathfrak{A})$ *with $\|\alpha - I\| < 2$. Then α lies in a norm-continuous one parameter subgroup of* Aut $(\mathfrak{A})$. *Such subgroups generate* $\text{Aut}_0(\mathfrak{A})$. *Each element of* $\text{Aut}_0(\mathfrak{A})$ *is π-inner.*

It follows from Theorem A that one has inclusions

$$\text{Aut}_0(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \varphi\text{-Inn}(\mathfrak{A}) \subseteq \sigma_\varphi(\mathfrak{A}) \subseteq \varphi\text{-Ext}(\mathfrak{A})$$

where φ is any faithful $*$-representation of $\mathfrak{A}$. Since $\text{Aut}_0(\mathfrak{A})$ contains an open ball with center I and radius 2 in Aut $(\mathfrak{A})$ it follows that each of $\pi(\mathfrak{A})$, φ-Inn $(\mathfrak{A})$, $\sigma_\varphi(\mathfrak{A})$, φ-Ext $(\mathfrak{A})$ is also an open subgroup of Aut $(\mathfrak{A})$. Hence they are all closed too.

Furthermore, $\mathrm{Aut}_0(\mathfrak{A})$ and $\pi(\mathfrak{A})$ are normal subgroups of $\mathrm{Aut}(\mathfrak{A})$, while in general, φ-Inn $(\mathfrak{A})$, $\sigma_\varphi(\mathfrak{A})$ and φ-Ext $(\mathfrak{A})$ are not [13, pp. 48–49 after Corollary 9].

The inner automorphism group Inn $(\mathfrak{A})$ of $\mathfrak{A}$ is contained in $\pi(\mathfrak{A})$, and is a normal subgroup of Aut $(\mathfrak{A})$. Generally, it is not true that $\mathrm{Aut}_0(\mathfrak{A})$ is contained in Inn $(\mathfrak{A})$ [13, p. 49].

Various examples in [13] show that all possible equalities and inequalities among the above inclusion relations actually occur. The interesting results are as follows:

THEOREM B [13, COROLLARY 9]. *If $\mathfrak{A}$ is a C^*-algebra with a faithful *-representation φ as a von Neumann algebra, then*

$$\mathrm{Inn}(\mathfrak{A}) = \mathrm{Aut}_0(\mathfrak{A}) = \pi(\mathfrak{A}) = \varphi\text{-}\mathrm{Inn}(\mathfrak{A})$$

and each element of $\mathrm{Aut}_0(\mathfrak{A})$ *lies on a norm-continuous one parameter subgroup of* Aut $(\mathfrak{A})$.

THEOREM C [13, IV, EXAMPLE b]. *Let $\mathfrak{A}$ be the C^*-algebra of compact operators on a separable Hilbert space with the identity operator I adjoined. (Thus every element $A \in \mathfrak{A}$ has the form $aI+K$, where $a \in C$ and K is a compact operator.) Then* $\mathrm{Aut}_0(\mathfrak{A}) = \mathrm{Aut}(\mathfrak{A}) = \pi(\mathfrak{A})$, *and $\mathfrak{A}$ admits noninner permanently weakly-inner automorphisms, i.e.* $\mathrm{Inn}(\mathfrak{A}) \subsetneq \pi(\mathfrak{A})$. *Since* $\mathrm{Inn}(\mathfrak{A}) = (I)$, *$\mathfrak{A}$ also provides an example where* $\mathrm{Aut}_0(\mathfrak{A}) \subsetneq \mathrm{Inn}(\mathfrak{A})$.

THEOREM D [13, IV, EXAMPLE d]. *Let X be a compact Hausdorff space, $\mathfrak{M}_n$ the C^*-algebra of all $(n \times n)$-matrices with complex entries and $\mathfrak{A} = C(X; \mathfrak{M}_n)$ the C^*-algebra of all continuous functions from X to $\mathfrak{M}_n$. (The norm in $\mathfrak{A}$ is the sup-norm, i.e. if $f \in C(X; \mathfrak{M}_n)$ then $\|f\| = \sup_{x \in X} \|f(x)\|$.) Then*

$$\mathrm{CF}(\mathfrak{A}) = \pi(\mathfrak{A}) = \varphi\text{-}\mathrm{Inn}(\mathfrak{A})$$

and

$$\mathrm{Aut}_0(\mathfrak{A}) \subseteq \mathrm{Inn}(\mathfrak{A}) \subseteq \pi(\mathfrak{A}).$$

All the possible equality and inequality relations among the inclusions in Theorem D can actually occur for a suitable choice of X. For example, if $X = I^n$, the n-dimensional cell, then [13, p. 57]

$$\mathrm{Aut}_0(\mathfrak{A}) = \mathrm{Inn}(\mathfrak{A}) = \pi(\mathfrak{A}).$$

If $X = U(n)/S^1$, where $U(n)$ is the unitary group in $\mathfrak{M}_n$, S^1 the circle group of diagonal matrices in $U(n)$, then [13, p. 67] if $n \geqq 3$,

$$\mathrm{Aut}_0(\mathfrak{A}) \neq \mathrm{Inn}(\mathfrak{A}) \neq \pi(\mathfrak{A}).$$

If $X = S^1$, the circle, then [13]

$$\mathrm{Aut}_0(\mathfrak{A}) \neq \mathrm{Inn}(\mathfrak{A}) = \pi(\mathfrak{A}).$$

If $X = \boldsymbol{RP}(3) = \boldsymbol{U}(2)/\boldsymbol{S}^1$ the real projective 3-space, then [13, p. 59] $\mathrm{Aut}_0(\mathfrak{A}) = \mathrm{Inn}(\mathfrak{A}) \neq \pi(\mathfrak{A})$.

Center-fixing automorphisms. Let $\mathfrak{A}$ be a C^*-algebra. As noted above the automorphisms of $\mathfrak{A}$ fixing the center of $\mathfrak{A}$ form a subgroup of $\mathrm{Aut}(\mathfrak{A})$ denoted by $\mathrm{CF}(\mathfrak{A})$. In this section some inclusion relations between $\mathrm{CF}(\mathfrak{A})$ and other subgroups of $\mathrm{Aut}(\mathfrak{A})$ will be discussed.

LEMMA 2.1. *Let $\mathfrak{A}$ be a C^*-algebra and φ a faithful $*$-representation of $\mathfrak{A}$ on a Hilbert space $\boldsymbol{H}$. Then*

$$\varphi\text{-Inn}(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A}). \qquad \square$$

As a consequence of Lemma 2.1 and the inclusions discussed above we have inclusions

$$\mathrm{Aut}_0(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \varphi\text{-Inn}(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A}) \subseteq \mathrm{Aut}(\mathfrak{A})$$

and

$$\mathrm{Inn}(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \varphi\text{-Inn}(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A}) \subseteq \mathrm{Aut}(\mathfrak{A}).$$

Thus $\mathrm{CF}(\mathfrak{A})$ contains an open ball in $\mathrm{Aut}(\mathfrak{A})$ with center I and radius 2. Thus $\mathrm{CF}(\mathfrak{A})$ is open and a closed subgroup of $\mathrm{Aut}(\mathfrak{A})$.

LEMMA 2.2. *If $\mathfrak{A}$ is a C^*-algebra then* $\mathrm{CF}(\mathfrak{A})$ *is a normal subgroup of* $\mathrm{Aut}(\mathfrak{A})$. $\square$

We will find it convenient to make use of tensor products of C^*-algebras in the remaining sections. We recall some of the relevant facts here.

Tensor products and cross-norms. If $\boldsymbol{H}$ and $\boldsymbol{K}$ are Hilbert spaces then we may introduce the Hilbert space tensor product $\boldsymbol{H} \otimes \boldsymbol{K}$ [2, p. 23], [19]. If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras with $*$-representations $\varphi: \mathfrak{A} \to \mathscr{L}(\boldsymbol{H})$, $\psi: \mathfrak{B} \to \mathscr{L}(\boldsymbol{K})$ then we denote by $\varphi \otimes \psi$ the algebraic tensor product $*$-representation $\varphi \otimes \psi: \mathfrak{A} \otimes \mathfrak{B} \to \mathscr{L}(\boldsymbol{H} \otimes \boldsymbol{K})$

$$(\varphi \otimes \psi)(A \otimes B)(\xi \otimes \eta) = \varphi(A)(\xi) \otimes \psi(B)(\eta),$$

where $A \in \mathfrak{A}$, $B \in \mathfrak{B}$, $\xi \in \boldsymbol{H}$, $\eta \in \boldsymbol{K}$.

THEOREM E (WULFSOHN [26, THEOREM 1]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, φ, φ' faithful $*$-representations of $\mathfrak{A}$ on $\boldsymbol{H}$, $\boldsymbol{H}'$ respectively, and ψ, ψ' faithful $*$-representations of $\mathfrak{B}$ on $\boldsymbol{K}$ and $\boldsymbol{K}'$ respectively. Then for any $A_i \in \mathfrak{A}$, $B_i \in \mathfrak{B}$*

$$\Big\| \sum_i \varphi(A_i) \otimes \psi(B_i) \Big\| = \Big\| \sum_i \varphi'(A_i) \otimes \psi'(B_i) \Big\|$$

where the norms are the norms in $\mathscr{L}(\boldsymbol{H} \otimes \boldsymbol{K})$ and $\mathscr{L}(\boldsymbol{H}' \otimes \boldsymbol{K}')$ respectively.

DEFINITION. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with faithful $*$-representations φ, ψ on $\boldsymbol{H}$ and $\boldsymbol{K}$ respectively. The completion of $\varphi(\mathfrak{A}) \otimes \psi(\mathfrak{B})$ in the norm topology on $\mathscr{L}(\boldsymbol{H} \otimes \boldsymbol{K})$ is denoted by $\mathfrak{A} \otimes^* \mathfrak{B}$.

REMARK. By Theorem E, $\mathfrak{A} \otimes^* \mathfrak{B}$ is independent of the choice of faithful *-representations φ and ψ.

In [27] Wulfsohn has shown that the cross-norm on $\mathfrak{A} \otimes^* \mathfrak{B}$ introduced above coincides with the α_0-norm of Turumaru [26].

III. **Locally inner automorphisms.** Let X be a compact Hausdorff space and $\boldsymbol{B}$ be a Banach algebra over $\boldsymbol{C}$. We denote by $C(X; \boldsymbol{B})$ the sup-norm algebra of all continuous functions $f: X \to \boldsymbol{B}$. If $\boldsymbol{B}$ is a C^*-algebra so is $C(X; \boldsymbol{B})$ and if $\boldsymbol{B}$ has an identity so does $C(X; \boldsymbol{B})$.

NOTATION. Let X be a compact Hausdorff space and $\boldsymbol{B}$ be a Banach algebra. If $B \in \boldsymbol{B}$ we denote by $\tilde{B}: X \to \boldsymbol{B}$ the constant function at $B \in \boldsymbol{B}$. This is clearly continuous and thus $\tilde{B} \in C(X; \boldsymbol{B})$. The function $\tilde{}: \boldsymbol{B} \to C(X; \boldsymbol{B})$ imbeds $\boldsymbol{B}$ in $C(X; \boldsymbol{B})$ in a natural way.

Let X be a compact Hausdorff space and $\boldsymbol{B}$ a C^*-algebra over $\boldsymbol{C}$. We define a function

$$e: C(X) \otimes \boldsymbol{B} \to C(X; \boldsymbol{B})$$

by

$$e\left(\sum_{i=1}^{n} f_i \otimes B_i\right)(x) = \sum_{i=1}^{n} f_i(x) B_i.$$

This function is continuous in the C^*-topology and extends to provide an isomorphism [18], [19], [21], [24]

$$e: C(X) \otimes^* \boldsymbol{B} \to C(X; \boldsymbol{B})$$

of C^*-algebras.

This isomorphism will prove a useful technical tool for us in this section.

We introduce, and fix, throughout this section, the following notation:

NOTATION. X is a compact Hausdorff space;

$\boldsymbol{H}$ is a Hilbert space over $\boldsymbol{C}$;

$\boldsymbol{B}$ denotes a C^*-algebra;

$C(X; \boldsymbol{B})$ is the sup-norm algebra of all continuous $f: X \to \boldsymbol{B}$;

$\mathfrak{A} = C(X; \boldsymbol{B})$;

Aut $(\mathfrak{A})$ is the group of automorphisms of $\mathfrak{A}$ with the uniform topology;

Aut $(\boldsymbol{B})$ is the group of all automorphisms of $\boldsymbol{B}$ with the uniform topology;

$C(X; \text{Aut}(\boldsymbol{B}))$ is the group of all continuous functions $X \to \text{Aut}(\boldsymbol{B})$, under pointwise multiplication. We equip $C(X; \text{Aut}(\boldsymbol{B}))$ with the compact-open topology. It is a topological group.

We turn now to the definition of locally-inner automorphisms of $\mathfrak{A}$. This will require some preparation.

NOTATION. $e: X \times C(X; \boldsymbol{B}) \to \boldsymbol{B}$ is the evaluation mapping given by $e(x, f) = f(x)$. If $x \in X$ we denote by

$$e_x: C(X; \boldsymbol{B}) \to \boldsymbol{B}$$

the mapping given by $e_x(f) = e(x, f)$.

PROPOSITION 3.1. *The evaluation mapping* $e\colon X \otimes C(X; \boldsymbol{B}) \to \boldsymbol{B}$ *is continuous.*

Proof. This follows from the fact that the norm topology contains the compact-open topology. See, for example, R. H. Fox, *Topologies for function spaces*, Bull. Amer. Math. Soc. **51** (1945), 429–432. □

LEMMA 3.2. *If* $\alpha \in \mathrm{CF}(\mathfrak{A})$ *and* $x \in X$, *then* $\alpha(\ker e_x) \subseteq \ker e_x$.

Proof. We will use the tensor product representation $C(X; \boldsymbol{B}) = C(X) \otimes^* \boldsymbol{B}$. Under this identification the evaluation mapping e_x is given by the extension of

$$e_x\colon C(X) \otimes \boldsymbol{B} \to \boldsymbol{B} \mid e_x\left(\sum_i f_i \otimes B_i\right) = \sum_i f_i(x) B_i.$$

Let $\mathfrak{M}$ be the maximal two sided ideal in $C(X)$ determined by $x \in X$, i.e. $\mathfrak{M} = \{f \in C(X) \mid f(x) = 0\}$. $\mathfrak{M}$ is again a C^*-algebra, albeit without identity. Thus $\mathfrak{M} \otimes^* \boldsymbol{B}$ is defined and $\mathfrak{M} \otimes^* \boldsymbol{B} \subset C(X) \otimes^* \boldsymbol{B}$ is a two sided ideal. Direct computation shows that $\mathfrak{M} \otimes^* \boldsymbol{B} = \ker e_x$.

Suppose $M_i \in \mathfrak{M}$, $B_i \in \boldsymbol{B}$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} \alpha\left(\sum_{i=1}^n M_i \otimes B_i\right) &= \sum_1^n \alpha(M_i \otimes B_i) = \sum_1^n \alpha[(M_i \otimes I)(I \otimes B_i)] \\ &= \sum_1^n \alpha(M_i \otimes I)\alpha(I \otimes B_i) \\ &= \sum_1^n (M_i \otimes I)\alpha(I \otimes B_i), \end{aligned}$$

since α is center-fixing and $Z(C(X) \otimes^* \boldsymbol{B}) = C(X) \otimes I$, and $\mathfrak{M} \otimes I \subset C(X) \otimes I$.

Applying the evaluation map to both sides of the above equality then gives

$$e_x \alpha\left(\sum_{i=1}^n M_i \otimes B_i\right) = \sum_{i=1}^n M_i(x) e_x(\alpha(I \otimes B_i)) = 0,$$

since $M_i \in \mathfrak{M}$. Since the elements of the form $\sum_{i=1}^n M_i \otimes B_i$, for finite $n > 0$, are dense in $\mathfrak{M} \otimes^* \boldsymbol{B} = \ker e_x$ and α, e_x are continuous with respect to the *-topology on $C(X) \otimes^* \boldsymbol{B}$, it follows that $e_x(\alpha(A)) = 0$ for any $A \in \mathfrak{M} \otimes^* \boldsymbol{B} = \ker e_x$, and thus $\alpha(\ker e_x) \subset \ker e_x$, as was to be shown. □

REMARK. Let S be a closed subspace of X. If $f\colon S \to \boldsymbol{C}$ is any continuous function then by Tietze's extension theorem there exists an extension of f, $F\colon X \to \boldsymbol{C}$.

The inclusion $S \hookrightarrow X$ induces by restriction a homomorphism of C^*-algebras (notice it is norm decreasing)

$$\rho\colon C(X; \boldsymbol{B}) \to C(S; \boldsymbol{B}).$$

Using the tensor product identifications we see that ρ is onto. For suppose $f_i \in C(S)$, $B_i \in \boldsymbol{B}$, $i=1,\ldots,n$. Choose extensions $F_i \in C(X)$ such that $F_i|_S=f_i$, $i=1,\ldots,n$. Then

$$\rho\left(\sum_i F_i \otimes B_i\right) = \sum_i f_i \otimes B_i.$$

Recall ρ is continuous with respect to the *-topology, $C(X; \boldsymbol{B})$ is complete, and the elements of the form $\sum_{i=1}^n f_i \otimes B_i$ are dense in $C(S) \otimes^* \boldsymbol{B}$. It follows that ρ is onto as claimed.

Thus given $f: S \to \boldsymbol{B}$ there exists an extension $F: X \to \boldsymbol{B}$. If F' is another such extension then $F-F'|_S: S \to \boldsymbol{B}$ is the constant function at $0 \in \boldsymbol{B}$. Thus for any $s \in S$ and $\alpha \in \mathrm{CF}\,(C(X; \boldsymbol{B}))$, $(F-F')(s)=0$. Therefore $\alpha(F)|_S=\alpha(F')|_S: S \to \boldsymbol{B}$.

Definition. If $\alpha \in \mathrm{CF}\,(\mathfrak{A})$ and S is a closed subspace of X define $\alpha|_S: C(S; \boldsymbol{B}) \to C(S; \boldsymbol{B})$ by

$$\alpha|_S\,(f)(x) = \alpha(F)(x)$$

where $F: X \to \boldsymbol{B}$ is any extension of f such that $F \in \mathfrak{A}$. This is well defined by the above remark.

Proposition 3.3. *Let X be a compact Hausdorff space, $S \subset X$ a closed subspace. Let $\boldsymbol{B}$ be a C^*-algebra, and $\alpha \in \mathrm{CF}\,(C(X; \boldsymbol{B}))$. Then*

$$\alpha|_S: C(S; \boldsymbol{B}) \to C(S; \boldsymbol{B}),$$

is a center-fixing automorphism of $C(S; \boldsymbol{B})$.

If $\rho: C(X; \boldsymbol{B}) \to C(S; \boldsymbol{B})$ is the homomorphism of C^*-algebras given by restriction then the diagram

$$\begin{array}{ccc} C(X; \boldsymbol{B}) & \xrightarrow{\ \alpha\ } & C(X; \boldsymbol{B}) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ C(S; \boldsymbol{B}) & \xrightarrow{\ \alpha|_S\ } & C(S; \boldsymbol{B}) \end{array}$$

is commutative. Thus

$$\alpha|_S: \mathrm{CF}\,(C(X; \boldsymbol{B})) \to \mathrm{CF}\,(C(S; \boldsymbol{B}))$$

is a continuous homomorphism of topological groups.

Proof. The verifications are all routine. □

We introduce now a subgroup of Aut $(\mathfrak{A})$ that will be of interest throughout the remainder of this paper.

Notation. If X is a topological space and $S \subset X$ is a subspace, then the closure of S in X is denoted by $\bar{S}$.

DEFINITION. Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra, and $\mathfrak{A}=C(X;\boldsymbol{B})$. If $\alpha\in \mathrm{CF}(\mathfrak{A})$ then α is locally-inner iff there exists an open covering $S_1,\ldots,S_N$ of X such that $\alpha|_{S_i}$ is inner, $i=1,\ldots,N$. The set of all locally inner automorphisms is denoted by loc-Inn $(\mathfrak{A})$.

PROPOSITION 3.4. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X;\boldsymbol{B})$. Then* loc-Inn $(\mathfrak{A})$ *is a subgroup of* Aut $(\mathfrak{A})$ *and*

$$\mathrm{Inn}\,(\mathfrak{A}) \subseteq \text{loc-Inn}\,(\mathfrak{A}) \subseteq \mathrm{CF}\,(\mathfrak{A}).$$

Proof. If $\alpha,\beta\in$ loc-Inn $(\mathfrak{A})$ then there exist open covers $\{S_1,\ldots,S_N\},\{T_1,\ldots,T_M\}$ of X with $\alpha|_{S_i}$, $i=1,\ldots,N$, $\beta|_{T_j}, j=1,\ldots,M$, inner. It then follows that $\beta^{-1}|_{T_j}$, $j=i,\ldots,M$, is inner. Let $V_{1,1},\ldots,V_{N,M}$ be the open cover of X given by $V_{i,j}=S_i\cap T_j$. It is immediate that $\alpha\beta^{-1}|_{V_{i,j}}$ is inner, and thus loc-Inn $(\mathfrak{A})$ is a subgroup of Aut $(\mathfrak{A})$. □

The main result of this section is:

THEOREM 3.5. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra, and $\mathfrak{A}=C(X;\boldsymbol{B})$. Then there exists a natural isomorphism of topological groups*

$$\tilde{}: \text{loc-Inn}\,(\mathfrak{A}) \to C(X;\mathrm{Aut}\,(\boldsymbol{B})).$$

The proof of Theorem 3.5 will occupy most of the remainder of this section.

DEFINITION. Let $\alpha\in\mathrm{CF}(\mathfrak{A})$ and $x\in X$. Define a function $\tilde{\alpha}_x\colon \boldsymbol{B}\to\boldsymbol{B}$ by $\tilde{\alpha}_x(B)=\alpha(\tilde{B})(x)$.

Thus for $\alpha\in\mathrm{CF}(\mathfrak{A})$ and $x\in X$, $\tilde{\alpha}_x$ is the function given by the composition

$$\boldsymbol{B}\xrightarrow{\sim} C(X;\boldsymbol{B})\xrightarrow{\alpha} C(X;\boldsymbol{B})\xrightarrow{e_x}\boldsymbol{B}.$$

LEMMA 3.6. *If $\alpha\in\mathrm{CF}(\mathfrak{A})$ and $x\in X$, then $\tilde{\alpha}_x\colon\boldsymbol{B}\to\boldsymbol{B}$ is a continuous *-preserving linear map.* □

LEMMA 3.7. *If $\alpha,\beta\in\mathrm{CF}(\mathfrak{A})$ and $x\in X$, then $((\beta\alpha)^\sim)_x=\tilde{\beta}_x\cdot\tilde{\alpha}_x$.*

Proof. For the sake of clarity in writing compositions of mappings we introduce the notation $\rho\colon\boldsymbol{B}\to C(X;\boldsymbol{B})$ for the imbedding $\tilde{}:\boldsymbol{B}\to C(X;\boldsymbol{B})$.

From the definition we then have $((\beta\cdot\alpha)^\sim)_x=e_x\beta\alpha\rho$ and $\tilde{\beta}_x\cdot\tilde{\alpha}_x=e_x\beta\rho e_x\alpha\rho$. Note that $e_x\rho\colon\boldsymbol{B}\to\boldsymbol{B}$ is the identity map, and thus $e_x\alpha\rho=e_x\rho e_x\alpha\rho$.

Let $B\in\boldsymbol{B}$. Then

$$\alpha\rho(B)-\rho e_x\alpha\rho(B)\in\ker e_x.$$

Since $\beta\in\mathrm{CF}(\mathfrak{A})$ it follows from Lemma 3.2 that $\beta\ker e_x\subset\ker e_x$. Therefore

$$\beta\alpha\rho(B)-\beta\rho e_x\alpha\rho(B)\in\ker e_x.$$

Hence

$$e_x\beta\alpha\rho(B)-e_x\beta\rho e_x\alpha\rho(B) = 0,$$

i.e.

$$e_x\beta\alpha\rho(B) = e_x\beta\rho e_x\alpha\rho(B)$$

for all $B \in \boldsymbol{B}$. From the definition of ~ this means

$$((\beta\alpha)^{\sim})_x(B) = \tilde{\beta}_x \cdot \tilde{\alpha}_x(B)$$

for all $B \in \boldsymbol{B}$, and thus $((\beta\alpha)^{\sim})_x = \tilde{\beta}_x \cdot \tilde{\alpha}_x$ as was to be shown. □

PROPOSITION 3.8. *If* $\alpha \in \mathrm{CF}(\mathfrak{A})$ *and* $x \in X$ *then* $\tilde{\alpha}_x \colon \boldsymbol{B} \to \boldsymbol{B}$ *is an automorphism of* $\boldsymbol{B}$.

Proof. It is a routine task to show that $\tilde{\alpha}_x$ is multiplicative. From Lemma 3.7 it follows that

$$(\tilde{\alpha}^{-1})_x = (\tilde{\alpha}_x)^{-1}.$$

Then the proposition follows from Lemma 3.6. □

DEFINITION. If $\alpha \in \mathrm{CF}(\mathfrak{A})$ let $\tilde{\alpha} \colon X \to \mathrm{Aut}(\boldsymbol{B})$ be the function defined by $\tilde{\alpha}(x) = \tilde{\alpha}_x$.

PROPOSITION 3.9. *Let* $X, \boldsymbol{B}, \mathfrak{A}$ *be as above. Then for any* $\alpha \in \mathrm{CF}(\mathfrak{A})$, $\tilde{\alpha} \colon X \to \mathrm{Aut}(\boldsymbol{B})$ *is strongly continuous.*

Proof. Let $\{x_\lambda \mid \lambda \in \Lambda\}$ be a convergent net in X with limit x. Then for any $B \in \boldsymbol{B}$, $\{(x_\lambda, \tilde{B}) \mid \lambda \in \Lambda\}$ is a convergent net in $X \times C(X; \boldsymbol{B})$ with limit $(x, \tilde{B})$. From the definition of $\tilde{\alpha}$ we have

$$\lim_{\lambda \in \Lambda} \tilde{\alpha}(x_\lambda)(B) = \lim_{\lambda \in \Lambda} e(x_\lambda, \alpha(\tilde{B})) = e(x, \alpha(\tilde{B})) = \tilde{\alpha}(x)(B),$$

since $e \colon X \times C(X; \boldsymbol{B}) \to \boldsymbol{B}$ is continuous. Thus for each $B \in \boldsymbol{B}$

$$\lim_{\lambda \in \Lambda} \tilde{\alpha}(x_\lambda)(B) = \tilde{\alpha}(x)(B).$$

Thus α takes convergent nets in X into strongly convergent nets in $\boldsymbol{B}$ and hence $\tilde{\alpha}$ is strongly continuous. □

Since the strong operator and uniform topologies of $\mathrm{Aut}(\boldsymbol{B})$ coincide when $\boldsymbol{H}$ is finite dimensional we obtain

COROLLARY 3.10. *Let* X *be a compact Hausdorff space,* $\boldsymbol{H}$ *a finite-dimensional Hilbert space,* $\boldsymbol{B} \subseteq \mathscr{L}(\boldsymbol{H})$ *and* $\mathfrak{A} = C(X; \boldsymbol{B})$. *Then for any* $\alpha \in \mathrm{CF}(\mathfrak{A})$, $\tilde{\alpha} \colon X \to \mathrm{Aut}(\boldsymbol{B})$ *is continuous in the uniform topology.* □

If the dimension of $\boldsymbol{H}$ is not finite we have not succeeded in showing $\tilde{\alpha} \colon X \to \mathrm{Aut}(\boldsymbol{B})$ is continuous for all $\alpha \in \mathrm{CF}(\mathfrak{A})$, in general. But, when X is separable and $\boldsymbol{B} = \mathscr{L}(\boldsymbol{H})$ then we may establish the continuity of $\tilde{\alpha}$ with the aid of the following theorem of Kallman [14].

THEOREM F (KALLMAN). *Let $\boldsymbol{R}$ be any von Neumann algebra, φ_n elements of the automorphism group of $\boldsymbol{R}$ $(n>0)$ such that $\|\varphi_n(T)-T\| \to 0$ $(n \uparrow \infty)$ for all $T \in \boldsymbol{R}$. Then $\|\varphi_n - I\| \to 0$ $(n \uparrow \infty)$.*

THEOREM 3.11. *Suppose that X is a separable compact Hausdorff space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ for a Hilbert space $\boldsymbol{H}$, and $\mathfrak{A}=C(X;\boldsymbol{B})$. If $\alpha \in \mathrm{CF}(\mathfrak{A})$ then $\tilde{\alpha}: X \to \mathrm{Aut}(\boldsymbol{B})$ is continuous in the uniform operator topology.*

Proof. As X is separable it will suffice to show that for each convergent *sequence* $\{x_n \mid x_n \in X\}$ with limit $x \in X$ we have

$$\lim_{n\to\infty} \|\tilde{\alpha}(x_n)-\tilde{\alpha}(x)\| = 0.$$

By replacing $\tilde{\alpha}$ by $\tilde{\alpha}(x)^{-1}\tilde{\alpha}$ we are reduced to considering the case where $\tilde{\alpha}(x)=I$. By Proposition 3.9 we have

$$\lim_{n\to\infty} \|\tilde{\alpha}(x_n)(T)-T\| = 0$$

for all $T \in \boldsymbol{B}$. Applying Kallman's theorem to the von Neumann algebra $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ now yields the desired conclusion. □

Attempts to extend Theorem 3.11 to arbitrary C^*-algebras would seem to depend on extending Kallman's theorem to a more general family of C^*-algebras than $\mathscr{L}(\boldsymbol{H})$.

CONJECTURE. Let X be a compact Hausdorff space and $\boldsymbol{H}$ a Hilbert space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A}=C(X;\boldsymbol{B})$. Then for any $\alpha \in \mathrm{CF}(\mathfrak{A})$, $\tilde{\alpha}: X \to \mathrm{Aut}(\boldsymbol{B})$ is continuous in the uniform topology.

While we are unable to settle the continuity of ~ in general we do have:

THEOREM 3.12. *Let X be a compact Hausdorff space, $\boldsymbol{H}$ a Hilbert space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A}=C(X;\boldsymbol{B})$. If $\alpha \in \mathrm{CF}(\mathfrak{A})$ then $\tilde{\alpha}: X \to \mathrm{Aut}(\boldsymbol{B})$ is continuous in the uniform operator topology iff $\alpha \in$ loc-Inn $(\mathfrak{A})$.*

REMARK. It follows from Theorem 3.11 and Corollary 3.10 that $\mathrm{CF}(\mathfrak{A})=$ loc-Inn $(\mathfrak{A})$ when the dimension of $\boldsymbol{H}$ is finite. We conjecture that this equality holds with no restriction on $\boldsymbol{H}$.

The proof of Theorem 3.12 will require some preparation.

NOTATION. $\mathscr{U}$ is the set of unitary elements of $\boldsymbol{B}$. It is a topological group when equipped with the induced topology from $\boldsymbol{B}$.

Let $\omega: \mathscr{U} \to \mathrm{Aut}(\boldsymbol{B})$ be the function defined by $\omega(U)=UBU^*$ for all $B \in \boldsymbol{B}$.

It follows from Kaplansky's theorem [3], [4] that $\omega: \mathscr{U} \to \mathrm{Aut}(\boldsymbol{B})$ is a surjection of abstract groups. The kernel of ω consists of those unitary elements U such that $UBU^*=B$ for all $B \in \boldsymbol{B}$, i.e. $\ker \omega = \mathscr{U} \cap Z(\boldsymbol{B})$. Since $Z(\boldsymbol{B})=\boldsymbol{C}\cdot I$ it follows that $\ker \omega = S^1 I$, where S^1 is the circle group.

Combining these observations with some calculations we obtain:

PROPOSITION 3.13. *The homomorphism* $\omega: \mathcal{U} \to \operatorname{Aut}(\boldsymbol{B})$ *is a continuous-open surjection and induces an isomorphism of topological groups* $\bar{\omega}: \mathcal{U}/S^1 \to \operatorname{Aut}(\boldsymbol{B})$.

Proof. The fact that ω is continuous is obvious. To see that ω is open one may employ Lemma 5 of [13] or the following more elementary computation due to C. L. Fefferman.

LEMMA. *Let* $U \in \mathcal{U}$. *Then there exists* $t \in S^1$ *such that*

$$\|tU-I\| \leqq 2\pi\|\omega(U)-I\|.$$

Proof of Lemma. Let $U \in \mathcal{U}$. By the spectral theorem we have the spectral representation

$$U = \int_0^{2\pi} e^{i\theta}\, dE_\theta.$$

Let $e^{i\theta_1}$, $e^{i\theta_2} \in \sigma(U)$. Let $\varepsilon > 0$ and choose ε-eigenvectors x, y corresponding to $e^{i\theta_1}$, $e^{i\theta_2}$ respectively. Where we mean that x is an ε-eigenvector if

$$|Ux-e^{i\theta_1}x| < \varepsilon|x|,$$

where $|\ |$ denotes the norm in $\boldsymbol{H}$. We may assume that $|x| = 1 = |y|$.

Let A be the linear operator on $\boldsymbol{H}$ defined by $Az = (z, x)y$, $z \in \boldsymbol{H}$. Then $Ax = y$, and

$$U^*x = U^{-1}x = e^{-i\theta_1}x+z$$

where $|z| \to 0$ as $\varepsilon \to 0$.

$$AU^*x = e^{-i\theta_1}Ax+Az = e^{-i\theta_1}y+Az,$$
$$UAU^*x = e^{-i\theta_1}Uy+w = e^{i(\theta_2-\theta_1)}y+w,$$

where $|w| \to 0$ as $\varepsilon \to 0$.

By definition

$$\begin{aligned} \|\omega(U)-I\| &= \sup_{\|B\|\leqq 1} \|UBU^*-B\| \\ &= \sup_{\|B\|\leqq 1} \sup_{|v|\leqq 1} |UBU^{-1}v-Bv|. \end{aligned}$$

Therefore

$$\begin{aligned} \|\omega(U)-I\| &\geqq |UAU^*x-Ax| \\ &= |e^{i(\theta_2-\theta_1)}Ax-Ax+w| \\ &\geqq |e^{i(\theta_2-\theta_1)}-1|\,|Ax|-|w| \\ &= |e^{i(\theta_2-\theta_1)}-1|-|w| \\ &= |e^{i\theta_2}-e^{i\theta_1}|\,|e^{-i\theta_1}|-|w| \\ &= |e^{i\theta_2}-e^{i\theta_1}|-|w|. \end{aligned}$$

Thus

$$(*) \qquad \|\omega(U)-I\| \geqq |e^{i\theta_2}-e^{i\theta_1}|-|w|.$$

Denote by $\Delta(U)$ the diameter of the spectrum of U, i.e.

$$\Delta(U) = \sup_{\exp(i\theta),\exp(i\theta')\in\sigma(U)} |e^{i\theta}-e^{i\theta'}|.$$

Taking the sup of both sides in (*) gives

$$\|\omega(U)-I\| \geqq \Delta(U)-\eta$$

for any $\eta>0$. Therefore

$$\|\omega(U)-I\| \geqq \Delta(U).$$

On the other hand, let $t=e^{i\theta_0}\in\sigma(U)$. Then

$$\begin{aligned}\|U-tI\| &= \left\| \int_0^{2\pi} (e^{i\theta}-e^{i\theta_0})\, dE_\theta \right\| \\ &\leqq 2\pi \sup_{\exp(i\theta)\in\sigma(U)} |e^{i\theta}-e^{i\theta_0}| \\ &\leqq 2\pi\Delta(U).\end{aligned}$$

Thus there exists $t\in\sigma(U)\subset S^1$, with

$$\begin{aligned}\|tU-I\| = \|U-tI\| &\leqq 2\pi\Delta(U) \\ &\leqq 2\pi\|\omega(U)-I\|. \qquad \square\end{aligned}$$

With the aid of this lemma we obtain the openness of ω as follows. Since ω is a homomorphism it suffices to show that ω is open at I. Let

$$N_\varepsilon = \{U\in\mathcal{U} \mid \|U-I\| < \varepsilon\}$$

be a basic open neighbourhood of $I\in\mathcal{U}$. We wish to exhibit a small open neighbourhood of $I\in\operatorname{Aut}(\boldsymbol{B})$ contained in $\omega(N_\varepsilon)$. Consider the open set

$$\mathcal{O}_\varepsilon = \{T\in\operatorname{Aut}(\boldsymbol{B}) \mid \|T-I\| < \varepsilon/3\pi\}.$$

Let $T\in\mathcal{O}_\varepsilon$. There exists $U\in\mathcal{U}$ such that $\omega(U)=T$, as we remarked above. By the Lemma above there exists $t\in S^1$ such that

$$\|tU-I\| \leqq 2\pi\|\omega(U)-I\| = 2\pi\|T-I\| < \varepsilon.$$

But $\omega(tU)=\omega(U)$. Therefore $T\in\omega(N_\varepsilon)$. Hence $\mathcal{O}_\varepsilon\subset\omega(N_\varepsilon)$ as required.

Thus the induced isomorphism $\bar{\omega}\colon \mathcal{U}/S^1\to\operatorname{Aut}(\boldsymbol{B})$ is an isomorphism of topological groups, where $\mathcal{U}/S^1$ is equipped with the quotient topology. $\square$

NOTATION. If X is a compact Hausdorff space and G is a topological group we denote by $C(X, G)$ the group of all continuous functions $X\to G$. The group operation is the pointwise product. We equip $C(X, G)$ with the compact-open topology making it into a topological group.

COROLLARY 3.14. *Let X be a compact Hausdorff space, $\boldsymbol{H}$ a Hilbert space and $\boldsymbol{B}=\mathcal{L}(\boldsymbol{H})$. Then ω induces a natural isomorphism of topological groups*

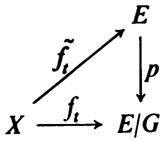
$$\omega_*\colon C(X; \mathcal{U}/S^1) \xrightarrow{\cong} C(X, \operatorname{Aut}(\boldsymbol{B})). \qquad \square$$

Recollections from algebraic topology. We assume that the reader is familiar with basic ideas of fiber bundle theory [11, pp. 39–41], [23, pp. 432–437], [24, Part I]. The following fundamental property of principal G-bundles will be used later.

COVERING HOMOTOPY THEOREM [23, THEOREM 7.2.6] OR [24, THEOREM 11.3]. *Let G be a topological group and (E, μ) a principal G-bundle. Let X be a topological space and suppose given*

(1) $f_t\colon X \to E/G$, $t \in [0, 1]$ *a homotopy class of mappings, and*

(2) $f_0\colon X \to E$ *a map such that* $p\tilde{f}_0 = f_0$. *Then there exists a homotopy* $\tilde{f}_t\colon X \to E$, $t \in [0, 1]$, *of* $\tilde{f}_0$, *such that the following diagram commutes for all* $t \in [0, 1]$, *i.e.* $p\tilde{f}_t = f_t$ *for all* $t \in [0, 1]$.



Let $\boldsymbol{H}$ be a (complex) Hilbert space, $\boldsymbol{B} = \mathscr{L}(\boldsymbol{H})$ and $\mathscr{U}$ the unitary elements in $\boldsymbol{B}$. Then

$$p\colon \mathscr{U} \to \mathscr{U}/S^1$$

is a principal S^1-bundle, where $S^1 = Z(\mathscr{U})$ acts on $\mathscr{U}$ by left translation [11, p. 41].

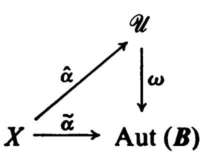
LEMMA 3.15. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra. Then an element $f \in C(X; \boldsymbol{B})$ is unitary iff $f(x)$ is unitary in $\boldsymbol{B}$ for all $x \in X$.* □

PROPOSITION 3.16. *Let X be a compact Hausdorff space, $\boldsymbol{H}$ a Hilbert space, $\boldsymbol{B} = \mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A} = C(X; \boldsymbol{B})$. Then $\alpha \in$ Aut $(\mathfrak{A})$ is inner iff there exists a map $\hat{\alpha}\colon X \to \mathscr{U}$ such that $\omega\hat{\alpha} = \tilde{\alpha}\colon X \to$ Aut $(\boldsymbol{B})$.*

Proof. Suppose that α is inner. Then from Lemma 3.15 it follows that there exists $\hat{\alpha} \in \mathfrak{A}$, $\hat{\alpha}\colon X \to \mathscr{U}$ such that $\alpha(f) = \hat{\alpha} f \hat{\alpha}^*$ for all $f \in C(X; \boldsymbol{B})$. Let $B \in \boldsymbol{B}$ and $x \in X$. Then from the definition of $\tilde{\alpha}$ we have

$$\tilde{\alpha}(x)(B) = \alpha(\tilde{B})(x) = \hat{\alpha}(x)\tilde{B}(x)\hat{\alpha}^*(x) = \hat{\alpha}(x)B\hat{\alpha}(x)^*.$$

Thus for each $x \in X$ the automorphism $\tilde{\alpha}(x) \in$ Aut $(\boldsymbol{B})$ agrees with the inner automorphism $B \to \hat{\alpha}(x)B\hat{\alpha}(x)^*$. Recalling our identification $\bar{\omega}\colon \mathscr{U}/S^1 \cong$ Aut $(\boldsymbol{B})$ this means that the diagram



commutes, i.e. $\omega\hat{\alpha} = \tilde{\alpha}$.

Conversely, suppose that there exists $\hat{\alpha}: X \to \mathscr{U}$ such that $\omega\hat{\alpha} = \tilde{\alpha}: X \to \operatorname{Aut}(\boldsymbol{B})$. Define an automorphism

$$\beta: \mathfrak{A} \to \mathfrak{A} \mid \beta(f)(x) = \hat{\alpha}(x)f(x)\hat{\alpha}(x)^*.$$

This is the inner automorphism of $\mathfrak{A}$ determined by $\hat{\alpha} \in C(X, \boldsymbol{B}) = \mathfrak{A}$.

We assert that $\beta = \alpha$. For suppose that $f \in C(X, \boldsymbol{B})$. Let $x \in X$. Then $e_x(f - (f(x))^\sim) = 0$. Therefore since $\alpha, \beta \in \mathrm{CF}(\mathfrak{A})$ it follows from Lemma 3.2 that

$$e_x(\alpha f - \alpha((f(x)^\sim))) = 0$$

and

$$e_x(\beta f - \beta((f(x))^\sim)) = 0.$$

Thus

$$(\alpha f)(x) = \alpha((f(x))^\sim)(x) = \tilde{\alpha}(x)(f(x)),$$
$$(\beta f)(x) = \beta((f(x))^\sim)(x) = \tilde{\beta}(x)(f(x)).$$

Now β being inner we have by direct computation (compare above) that

$$\tilde{\beta}(x)(f(x)) = \hat{\alpha}(x)f(x)\hat{\alpha}(x)^*.$$

But by hypothesis

$$\hat{\alpha}(x)f(x)\hat{\alpha}(x)^* = \tilde{\alpha}(x)(f(x)).$$

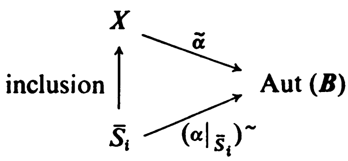
Therefore, combining these equalities yields

$$(\alpha f)(x) = \tilde{\alpha}(x)(f(x)) = \hat{\alpha}(x)f(x)\hat{\alpha}(x)^*$$
$$= \tilde{\beta}(x)(f(x)) = (\beta f)(x).$$

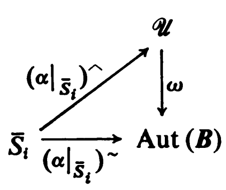
Hence $(\alpha f)(x) = (\beta f)(x)$ for all $x \in X$ and thus $\alpha f = \beta f$. Since $f \in C(X, \boldsymbol{B})$ was arbitrary it follows that $\alpha f = \beta f = \hat{\alpha} f \hat{\alpha}^*$ for all $f \in C(X, \boldsymbol{B})$ and hence α is inner. □

Proof of Theorem 3.12. Suppose that $\alpha \in \mathrm{CF}(\mathfrak{A})$ is locally inner. Let $S_1, \ldots, S_N$ be an open cover of X such that $\alpha|_{S_i}$ is inner, $i = 1, \ldots, N$.

From Proposition 3.3 it follows that we have a commutative diagram



and thus $\tilde{\alpha}|_{\bar{S}_i} = (\alpha|_{\bar{S}_i})^\sim$. Since $\alpha|_{\bar{S}_i}$ is an inner automorphism of $C(\bar{S}_i, \boldsymbol{B})$ it follows from Proposition 3.16 that there is a continuous function $(\alpha|_{\bar{S}_i})^\wedge: \bar{S}_i \to \mathscr{U}$ such that the diagram



commutes. Thus $(\alpha|_{\bar{S}_i})^{\sim}: \bar{S}_i \to \mathrm{Aut}(\boldsymbol{B})$ is the composition of two continuous maps and hence is continuous. Therefore $\tilde{\alpha}|_{S_i}: S_i \to \mathrm{Aut}(\boldsymbol{B})$ is continuous. Since $S_1, \ldots, S_N$ is an open cover of X and $\tilde{\alpha}|_{S_1}, \ldots, \tilde{\alpha}|_{S_N}$ are continuous, it follows that $\tilde{\alpha}$ is continuous.

Conversely suppose that $\tilde{\alpha}: X \to \mathrm{Aut}(\boldsymbol{B})$ is continuous. Under the identification $\bar{\omega}: \mathscr{U}/S^1 = \mathrm{Aut}(\boldsymbol{B})$ the mapping $\omega: \mathscr{U} \to \mathrm{Aut}(\boldsymbol{B})$ becomes a locally trivial fiber bundle. Thus there exists an open cover $\{\mathscr{O}_\lambda \mid \lambda \in \Lambda\}$ of $\mathrm{Aut}(\boldsymbol{B})$ and continuous functions $s_\lambda: \mathscr{O}_\lambda \to \mathscr{U}$ such that $\omega s_\lambda = 1: \mathscr{O}_\lambda \to \mathscr{O}_\lambda$.

Since $\tilde{\alpha}$ is continuous $\{\tilde{\alpha}^{-1}\mathscr{O}_\lambda \mid \lambda \in \Lambda\}$ is an open cover of X. Since X is compact Hausdorff we may select a finite open covering $S_1, \ldots, S_N$ of X with $\bar{S}_i \subset \tilde{\alpha}^{-1}\mathscr{O}_{\lambda_i}$, $i = 1, \ldots, N$, for some $\lambda_1, \ldots, \lambda_N \in \Lambda$. (This process is called shrinking the cover.) Therefore $\tilde{\alpha}(\bar{S}_i) \subset \mathscr{O}_{\lambda_i}$.

Define $\hat{\alpha}_i: \bar{S}_i \to \mathscr{U}$ by $\hat{\alpha}_i = s_{\lambda_i}\tilde{\alpha}|_{\bar{S}_i}$. Then

$$(\alpha|_{\bar{S}_i})^{\sim} = \tilde{\alpha}|_{\bar{S}_i} = \omega(s_{\lambda_i}\tilde{\alpha}|_{\bar{S}_i}) = \omega\hat{\alpha}_i.$$

Hence by Proposition 3.16 $\alpha|\bar{S}_i$ is inner. Therefore $S_1, \ldots, S_n$ is an open cover of X such that $\alpha|_{\bar{S}_1}, \ldots, \alpha|_{\bar{S}_N}$ are inner and hence α is locally inner. □

Proof of Theorem 3.5. It follows from Theorem 3.12 that we may define a function

$$^{\sim}: \text{loc-Inn}(\mathfrak{A}) \to C(X, \mathrm{Aut}(\boldsymbol{B}))$$

by $\alpha \to \tilde{\alpha}$.

To show that $^{\sim}$ is a homomorphism of abstract groups recall that the group operation on $C(X, \mathrm{Aut}(\boldsymbol{B}))$ is pointwise multiplication. Thus we need only show that for each $x \in X$, $\alpha, \beta \in \text{loc-Inn}(\mathfrak{A})$ that $((\beta\alpha)^{\sim})(x) = \tilde{\beta}(x)\tilde{\alpha}(x)$. But since $\text{loc-Inn}(\mathfrak{A}) \subset \mathrm{CF}(\mathfrak{A})$, this follows from Lemma 3.7.

The continuity and openness of $^{\sim}$ are exercises in the use of the compact-open topology.

To prove that $^{\sim}$ is an isomorphism of abstract groups we define a function

$$^{\wedge}: C(X, \mathrm{Aut}(\boldsymbol{B})) \to \text{loc-Inn}(\mathfrak{A})$$

as follows.

Let $\varphi \in C(X, \mathrm{Aut}(\boldsymbol{B}))$. For any $f \in C(X, \boldsymbol{B})$ define $\hat{\varphi}(f)(x) = (\varphi(x))(f(x))$, for any $x \in X$. Note that $\hat{\varphi}f$ is the composite

$$X \xrightarrow{\text{diag}} X \times X \xrightarrow{f \times \varphi} \boldsymbol{B} \times \mathrm{Aut}(\boldsymbol{B}) \xrightarrow{e} \boldsymbol{B}$$

and hence is continuous (compare Proposition 3.1). Thus $\hat{\varphi}$ defines a function $\hat{\varphi}: C(X, \boldsymbol{B}) \to C(X, \boldsymbol{B})$. Direct computation shows that $\hat{\varphi} \in \mathrm{Aut}(\mathfrak{A})$. If $f \in \boldsymbol{Z}(\mathfrak{A})$, then there exists a continuous function $g: X \to \boldsymbol{C}$ such that $f = gI$. Thus for any $x \in X$

$$\begin{aligned}(\hat{\varphi}(f))(x) &= \varphi(x)(f(x)) = \varphi(x)(g(x)I) \\ &= g(x)\varphi(x)I = g(x)I = f(x).\end{aligned}$$

Therefore $\hat{\varphi}(f)=f$ and $\hat{\varphi} \in \mathrm{CF}(\mathfrak{A})$.

For any $B \in \boldsymbol{B}$ and $x \in X$ we have

$$\hat{\varphi}^{\sim}(x)(B) = \hat{\varphi}(\tilde{B})(x) = \varphi(x)(\tilde{B}(x)) \\ = \varphi(x)(B).$$

and therefore

$$\hat{\varphi}^{\sim} = \varphi \colon X \to \mathrm{Aut}(\boldsymbol{B}).$$

Hence $\hat{\varphi}^{\sim}$ is a continuous function and therefore $\hat{\varphi} \in$ loc-Inn $(\mathfrak{A})$ by Theorem 3.12. This defines $\hat{}$.

Routine calculation shows that

$$\hat{}\colon C(X, \mathrm{Aut}(\boldsymbol{B})) \to \text{loc-Inn}(\mathfrak{A})$$

is a homomorphism of abstract groups.

We have already seen that $\hat{\varphi}^{\sim}=\varphi$ for any $\varphi \in C(X, \mathrm{Aut}(\boldsymbol{B}))$. If $\beta \in$ loc-Inn $(\mathfrak{A})$ then for any $f \in C(X, \boldsymbol{B})$, $x \in X$,

$$e_x(f-(f(x))^{\sim}) = 0.$$

Thus (compare the proof of Proposition 3.16) by Lemma 3.2,

$$(\beta f)(x) = \beta(f(x)^{\sim})(x).$$

Therefore from the definitions of $\hat{}$ and $\tilde{}$ we have for any $\alpha \in$ loc-Inn $(\mathfrak{A})$

$$(\tilde{\alpha}\hat{}f)(x) = \tilde{\alpha}\hat{}(f(x))^{\sim}(x) = \tilde{\alpha}(x)((f(x))^{\sim}(x)) \\ = \tilde{\alpha}(x)(f(x)) = \alpha(f(x))^{\sim}(x) \\ = (\alpha(f))(x).$$

Therefore $\tilde{\alpha}\hat{}f=\alpha f$ and hence $\tilde{\alpha}\hat{}=\alpha$. Therefore $\hat{}$ and $\tilde{}$ are inverse isomorphisms of abstract groups and the result follows. □

COROLLARY 3.17. *Let X be a compact Hausdorff space, $\boldsymbol{H}$ a Hilbert space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A}=C(X, \boldsymbol{B})$. Then there exists a natural isomorphism of topological groups* $\tilde{}$: loc-Inn $(\mathfrak{A})=C(X, \mathscr{U}/S^1)$; *where $\mathscr{U} \subset \boldsymbol{B}$ is the unitary group and $S^1=Z(\mathscr{U})$.*

Proof. This follows from Theorem 3.5 and Proposition 3.13. □

COROLLARY 3.18. *Let X be a separable compact Hausdorff space, $\boldsymbol{H}$ a Hilbert space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A}=C(X; \boldsymbol{B})$. Then* CF $(\mathfrak{A})=$ loc-Inn $(\mathfrak{A})$.

Proof. This follows from Theorems 3.11 and 3.12. □

IV. **A special case.** In this section we will apply the results of the previous section to the algebra $C(X; \mathscr{L}(\boldsymbol{H}))$ when $\boldsymbol{H}$ is infinite dimensional. We shall need the following remarkable theorem of Kuiper.

THEOREM G (KUIPER [16]). *Let H be an infinite-dimensional Hilbert space and $\mathcal{U}$ the unitary group of H. Then $\mathcal{U}$ is contractible in the uniform topology.*

NOTATION. If X and Y are topological spaces we denote by $[X, Y]$ the set of homotopy classes of maps $f: X \to Y$. The homotopy class of f is denoted by $[f]$.

If G is a topological group then $[X, G]$ may be given the structure of discrete group by pointwise multiplication (of representatives) of homotopy classes.

Recollection from algebraic topology. If π is an abelian group and n is a positive integer, an Eilenberg-Mac Lane space of type (π, n) is a topological space $K(\pi, n)$ such that

$$\begin{aligned} \pi_i(K(\pi, n)) &= \pi \quad \text{if } i = n, \\ &= 0 \quad \text{if } i \neq n, \end{aligned}$$

where $\pi_i(\ )$ denotes the ith-homotopy group [23].

For example, the circle, S^1, is a $K(Z, 1)$-space, where Z is the additive group of integers.

The spaces $K(\pi, n)$ have a natural abelian group structure and thus for any space X, $[X, K(\pi, n)]$ is an abelian group. In fact for any compact space X there is a natural isomorphism [23] of abelian groups,

$$[X, K(\pi, n)] = \check{H}^n(X; \pi),$$

where $\check{H}^n(X, \pi)$ denotes the nth Čech cohomology group of X with coefficients in π. If X is a "nice" space (for example, a cell complex with the weak topology, i.e. a cw-complex) then

$$[X, K(\pi, n)] = H^n(X; \pi),$$

where $H^n(X; \pi)$ denotes the nth singular cohomology group of X with coefficients in π. More precisely, for such nice spaces, Čech and singular cohomologies are naturally isomorphic [28].

If G is a topological group and $p: E \to E/G$ is a principal G-bundle then there is an exact homotopy sequence [23, Theorem 7.2.10], [24, Theorem 17.4],

$$\cdots \longrightarrow \pi_i(G) \longrightarrow \pi_i(E) \xrightarrow{p_*} \pi_i(E/G) \xrightarrow{\partial} \pi_{i-1}(G) \longrightarrow \cdots.$$

If E is contractible then $\pi_i(E) = 0$ for all $i \geqq 0$, and hence

$$\partial: \pi_i(E/G) \to \pi_{i-1}(G)$$

is an isomorphism for all $i > 0$.

We may apply these considerations to the principal S^1-bundle $\mathcal{U} \to \mathcal{U}/S^1$. Since we have assumed H to be infinite dimensional $\mathcal{U}$ is contractible by the theorem of Kuiper. Therefore

$$\pi_i(\mathcal{U}/S^1) = \pi_{i-1}(S^1)$$

for all $i \geqq 1$. Since S^1 is a $K(Z, 1)$-space we obtain

$$\pi_i(\mathscr{U}/S^1) = Z \quad \text{if } i = 2,$$
$$= 0 \quad \text{if } i \neq 2.$$

Thus $\mathscr{U}/S^1$ is an Eilenberg-Mac Lane space of type $(Z, 2)$.

NOTATION. Let X be a topological space and G be a topological group. Denote by $C_0(X; G)$ the subgroup of $C(X; G)$ consisting of the null-homotopic maps. If X is compact Hausdorff then $C_0(X; G)$ is just the identity component of $C(X; G)$ with the compact open topology.

$C_0(X; G)$ is a normal subgroup of $C(X; G)$ and there is a natural isomorphism of discrete groups,

$$C(X; G)/C_0(X; G) = [X, G] \quad [23].$$

THEOREM 4.1. *Let X be a compact Hausdorff space, H an infinite-dimensional Hilbert space, $B = \mathscr{L}(H)$ and $\mathfrak{A} = C(X; B)$. Then there is a natural isomorphism of groups* loc-Inn $(\mathfrak{A})$/Inn $(\mathfrak{A}) = \check{H}^2(X; Z)$. *If moreover X is separable, we have the isomorphism of groups*

$$\text{loc-Inn}\,(\mathfrak{A})/\text{Inn}\,(\mathfrak{A}) = \check{H}^2(X; Z) = \text{CF}\,(\mathfrak{A})/\text{Inn}\,(\mathfrak{A}).$$

Proof. By 3.18, loc-Inn $(\mathfrak{A}) = C(X; \mathscr{U}/S^1)$. By Proposition 3.16,

$$\text{Inn}\,(\mathfrak{A}) = p_* C(X; \mathscr{U}).$$

Now we assert that

$$p_* C(X; \mathscr{U}) = C_0(X; \mathscr{U}/S^1)$$

where $C_0(X; \mathscr{U}/S^1)$ is the subgroup of $C(X; \mathscr{U}/S^1)$ consisting of the null-homotopic maps;

For suppose $\varphi: X \to \mathscr{U}/S^1$ is null homotopic. Choose a homotopy $\varphi_t: X \to \mathscr{U}/S^1$, $t \in [0, 1]$, with $\varphi_1 = \varphi$ and $\varphi_0 =$ the constant map at $I \in \mathscr{U}/S^1$. Let $\bar{\varphi}_0: X \to \mathscr{U}$ be the constant map at $I \in \mathscr{U}$. Clearly $p\bar{\varphi}_0 = \varphi_0$. Therefore by the covering homotopy theorem there exists a homotopy $\bar{\varphi}_t: X \to \mathscr{U}$ with $p\bar{\varphi}_t = \varphi_t$ for all $t \in [0, 1]$. Let $\bar{\varphi} = \bar{\varphi}_1$. Then $p\bar{\varphi} = \varphi$ and hence $\varphi \in p_* C(X; \mathscr{U})$. Thus $p_* C(X; \mathscr{U}) \supset C_0(X; \mathscr{U}/S^1)$.

Conversely if $\varphi \in p_* C(X; \mathscr{U})$ then we may choose $\bar{\varphi} \in C(X; \mathscr{U})$ with $p\bar{\varphi} = \varphi$. Since $\mathscr{U}$ is contractible by Kuiper's theorem, it follows that $\bar{\varphi}$ is null homotopic. Let $\bar{\varphi}_t: X \to \mathscr{U}$ be a homotopy with $\bar{\varphi}_0 =$ constant map at $I \in \mathscr{U}$ and $\bar{\varphi}_1 = \bar{\varphi}$. Then $\varphi_t = p\bar{\varphi}_t$ is a homotopy from φ to the constant map at $I \in \mathscr{U}/S^1$. Thus φ is null homotopic and hence $p_* C(X; \mathscr{U}) \subset C_0(X; \mathscr{U})$. Combining this with 3.16 and 3.17 we obtain

$$\text{loc-Inn}\,(\mathfrak{A})/\text{Inn}\,(\mathfrak{A}) = C(X; \mathscr{U}/S^1)/C_0(X; \mathscr{U}/S^1)$$
$$= [X, \mathscr{U}/S^1].$$

Since $\mathscr{U}/S^1$ is a $K(Z, 2)$-space this yields

$$\text{loc-Inn}\,(\mathfrak{A})/\text{Inn}\,(\mathfrak{A}) = \check{H}^2(X; Z)$$

as claimed. Using 3.18 we obtain the case where X is separable. $\square$

REMARK. E. C. Lance has shown [18] that when X is separable $\pi(\mathfrak{A})/\mathrm{Inn}\,(\mathfrak{A}) = \hat{H}^2(X; \mathbf{Z})$. Thus we obtain

COROLLARY 4.2. *If X is a separable compact Hausdorff space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ for an infinite-dimensional Hilbert space $\boldsymbol{H}$, and $\mathfrak{A}=C(X; \boldsymbol{B})$, then*

$$\pi(\mathfrak{A}) = \mathrm{CF}\,(\mathfrak{A}) = \text{loc-Inn}\,(\mathfrak{A}). \quad \square$$

Also,

THEOREM 4.3. *If X is a separable compact Hausdorff space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ for an infinite-dimensional Hilbert space $\boldsymbol{H}$, and $\mathfrak{A}=C(X; \boldsymbol{B})$ then* $\mathrm{Aut}_0\,(\mathfrak{A})=\mathrm{Inn}\,(\mathfrak{A})$ *where* $\mathrm{Aut}_0\,(\mathfrak{A})$ *is the identity component of* $\mathrm{Aut}\,(\mathfrak{A})$.

Proof. Let us denote by $C_0(X; \mathcal{U}/S^1)$ and $\mathrm{CF}_0\,(\mathfrak{A})$ the identity components of topological groups $C(X; \mathcal{U}/S^1)$ and $\mathrm{CF}\,(\mathfrak{A})$, respectively. First we want to show that

$$C_0(X; \mathcal{U}/S^1) = \mathrm{Inn}\,(\mathfrak{A}). \tag{1}$$

By 3.16 we know $\mathrm{Inn}\,(\mathfrak{A})=p_*C(X; \mathcal{U})$. Also in the proof of 4.1 we saw $p_*C(X; \mathcal{U})$ is null homotopic to $C(X; \mathcal{U}/S^1)$. But it is well known that [23] $f: X\to \mathcal{U}/S^1$ is null homotopic if and only if f belongs to the identity component of $C(X; \mathcal{U}/S^1)$, where $C(X; \mathcal{U}/S^1)$ is equipped with the compact-open topology. Thus $\mathrm{Inn}\,(\mathfrak{A})= C_0(X; \mathcal{U}/S^1)$, obtaining (1).

Next we note

$$\mathrm{CF}_0\,(\mathfrak{A}) = \mathrm{Aut}_0\,(\mathfrak{A}). \tag{2}$$

For by [13] and the remark in §II we have

$$\mathrm{Aut}_0\,(\mathfrak{A}) \subseteq \pi(\mathfrak{A}) \subseteq \mathrm{CF}\,(\mathfrak{A}) \subseteq \mathrm{Aut}\,(\mathfrak{A}).$$

So $\mathrm{CF}\,(\mathfrak{A})$ is open, and $\mathrm{Aut}_0\,(\mathfrak{A})=\mathrm{CF}_0\,(\mathfrak{A})$.

Hence we only have to show $\mathrm{Inn}\,(\mathfrak{A})=\mathrm{CF}_0\,(\mathfrak{A})$ in order to complete the proof of the theorem. But by Corollaries 3.17 and 3.18 we know that

$$C(X; \mathcal{U}/S^1) = \text{loc-Inn}\,(\mathfrak{A}) = \mathrm{CF}\,(\mathfrak{A}).$$

Hence $\mathrm{CF}_0\,(\mathfrak{A})=C_0(X; \mathcal{U}/S^1)$. But by (1), we obtain

$$\mathrm{CF}_0\,(\mathfrak{A}) = C_0(X; \mathcal{U}/S^1) = \mathrm{Inn}\,(\mathfrak{A}). \quad \square$$

REMARK. Theorem 4.3 has been obtained by Lance also, see [18].

More generally, when X is not separable we have the following:

If $\mathfrak{A}$ is a C^*-algebra recall that $\mathrm{Aut}_0\,(\mathfrak{A})$ is the identity component of $\mathrm{Aut}\,(\mathfrak{A})$ in the norm topology.

PROPOSITION 4.4. *Let X be a compact Hausdorff space, $\boldsymbol{H}$ an infinite-dimensional Hilbert space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A}=C(X; \boldsymbol{B})$. Then* $\mathrm{Inn}\,(\mathfrak{A})\subseteq\mathrm{Aut}_0\,(\mathfrak{A})$.

Proof. By Proposition 3.15 and Proposition 3.14 ~ induces an isomorphism of topological groups (compare with the proof of Theorem 4.1)

$$\tilde{}: \operatorname{Inn}(\mathfrak{A}) \cong p_* C(X; \mathscr{U}).$$

By Kuiper's theorem $\mathscr{U}$ is contractible and hence $C(X; \mathscr{U})$ is connected [22]. Therefore $p_* C(X; \mathscr{U})$ is connected and hence $\operatorname{Inn}(\mathfrak{A})$ is connected. Thus we must have $\operatorname{Inn}(\mathfrak{A}) \subseteq \operatorname{Aut}_0(\mathfrak{A})$. □

REMARK. Proposition 4.4 is in striking contrast with the case when dim $\boldsymbol{H}$ is finite. In [13, Example d] it is shown that $\operatorname{Aut}_0(\mathfrak{A}) \subsetneq \operatorname{Inn}(\mathfrak{A})$ when dim $\boldsymbol{H}$ is finite.

CONJECTURE. If X is a compact Hausdorff space, $\boldsymbol{H}$ an infinite-dimensional Hilbert space, $\boldsymbol{B} = \mathscr{L}(\boldsymbol{B})$ and $\mathfrak{A} = C(X; \boldsymbol{B})$ then $\operatorname{Inn}(\mathfrak{A}) = \operatorname{Aut}_0(\mathfrak{A})$.

This conjecture is closely related to (and would follow from) the continuity of $\tilde{\alpha}$.

COROLLARY 4.5. *Let $\boldsymbol{H}$ be an infinite-dimensional Hilbert space, $\boldsymbol{B} = \mathscr{L}(\boldsymbol{H})$, and G a finitely generated abelian group. Then there exists a separable compact Hausdorff space X such that* $G = \mathrm{CF}(\mathfrak{A})/\operatorname{Inn}(\mathfrak{A}) = \text{loc-Inn}(\mathfrak{A})/\operatorname{Inn}(\mathfrak{A})$, *where* $\mathfrak{A} = C(X; \boldsymbol{B})$.

Proof. According to Theorem 4.1 we need only construct a separable compact Hausdorff space X with $\check{H}^2(X; \boldsymbol{Z}) = G$. However this is well-known algebraic topology [23, Example C.6, p. 206]. □

COROLLARY 4.6. *Let X be a compact Hausdorff space, $\boldsymbol{H}$ an infinite-dimensional Hilbert space, $\boldsymbol{B} = \mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A} = C(X; \boldsymbol{B})$. Then* loc-Inn $(\mathfrak{A})/\operatorname{Inn}(\mathfrak{A})$ *is abelian. If moreover X is separable then* $\mathrm{CF}(\mathfrak{A})/\operatorname{Inn}(\mathfrak{A})$ *is always an abelian group.*

Proof. $\check{H}^2(X; \boldsymbol{Z})$ is always abelian. □

REMARK. Corollary 4.6 is in striking contrast to the situation when $\boldsymbol{H}$ is finite dimensional. For in the finite-dimensional case $\text{loc-Inn}(\mathfrak{A}) = \mathrm{CF}(\mathfrak{A})$ and the examples constructed in [13] show that $\mathrm{CF}(\mathfrak{A})/\operatorname{Inn}(\mathfrak{A})$ need not be abelian.

V. **Miscellaneous results.** In this section we collect various miscellaneous results concerning automorphisms of the C^*-algebras $C(X; \boldsymbol{B})$.

The $C(X)$-module structure. Let X be a compact Hausdorff space and $\boldsymbol{B}$ a C^*-algebra. Denote by $C(X)$ the C^*-algebra $C(X; \boldsymbol{C})$. There is a natural map

$$\mu: C(X) \otimes^* C(X; \boldsymbol{B}) \to C(X; \boldsymbol{B})$$

given by the *-linear continuous extension of the mapping

$$[\mu(f \otimes g)](x) = f(x) \cdot g(x) \in \boldsymbol{B}$$

where $f \in C(X)$ and $g \in C(X; \boldsymbol{B})$. We denote $\mu(f \otimes g)$ by $f \circ g$. This provides $C(X; \boldsymbol{B})$ with the structure of an (continuous) algebra over the algebra $C(X)$.

PROPOSITION 5.1. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra with $Z(\boldsymbol{B}) = \boldsymbol{C} \cdot I$. An automorphism of $C(X; \boldsymbol{B})$ is center-fixing iff it is an automorphism of $C(X)$-algebras.*

Proof. Suppose $\alpha \in \mathrm{CF}\,(C(X;\boldsymbol{B}))$. Let $f \in C(X)$, $g \in C(X;\boldsymbol{B})$. Then

$$\alpha(f \circ g) = \alpha((f \circ I)g).$$

Since $f \circ I \in \boldsymbol{Z}(C(X;\boldsymbol{B}))$ we obtain in addition

$$\alpha(f \circ g) = \alpha(f \circ I)\alpha(g) = (f \circ I)\alpha(g) = f \circ \alpha(g)$$

and thus α is an automorphism of $C(X)$-modules. Since it is also a C^*-algebra isomorphism it is an isomorphism of $C(X)$-algebras.

Next suppose that α is an automorphism of $C(X)$-algebras. Then α is an automorphism of the C^*-algebra $C(X;\boldsymbol{B})$. Let $f \in \boldsymbol{Z}(C(X;\boldsymbol{B}))$, then there exists $g: X \to \boldsymbol{C}$ such that $f = g \circ I$ and hence

$$\alpha(f) = \alpha(g \circ I) = g \circ \alpha(I) = g \circ I = f$$

and α is center-fixing as required. $\square$

Carefully ideal preserving automorphisms.

DEFINITION. Let $\mathfrak{A}$ be a C^*-algebra. An automorphism α of $\mathfrak{A}$ is said to be *ideal preserving* iff $\alpha(J) \subset J$ for every closed two sided ideal J of $\mathfrak{A}$.

An automorphism α of $\mathfrak{A}$ is said to be *carefully ideal preserving* iff $\alpha(J) = J$ for each closed two sided ideal J in $\mathfrak{A}$.

In this subsection we study the relation between center-fixing and ideal preserving automorphisms of the C^*-algebras $C(X;\boldsymbol{B})$.

We will use portions of the theory of §III.

NOTATION. Let $\mathfrak{A}$ be a C^*-algebra. Denote by $\tau(\mathfrak{A})$ the set of all ideal preserving automorphisms of $\mathfrak{A}$. Note that $\tau(\mathfrak{A})$ is only a subsemigroup of Aut $(\mathfrak{A})$.

Denote by $\tau_0(\mathfrak{A})$ the set of all carefully ideal preserving automorphisms of $\mathfrak{A}$. Note that $\tau_0(\mathfrak{A})$ is a subgroup of Aut $(\mathfrak{A})$ [17].

Let X be a compact Hausdorff space and $\boldsymbol{B}$ a C^*-algebra. Let $J \subset \boldsymbol{B}$ be a closed two sided ideal in $\boldsymbol{B}$ and $x \in X$ a fixed point. Define $J(x)$ in $C(X;\boldsymbol{B})$ by

$$J(x) = \{f \in C(X;\boldsymbol{B}) \mid f(x) \in J\}.$$

Then one readily checks that $J(x)$ is a closed two sided ideal in $C(X;\boldsymbol{B})$. These ideals may be used to describe the general form of the closed two sided ideals in $C(X;\boldsymbol{B})$. For the proof of the next theorem we refer to [5], [15].

THEOREM H (I. KAPLANSKY). *Let X be a compact Hausdorff space and $\boldsymbol{B}$ a C^*-algebra. If $J \subset C(X;\boldsymbol{B})$ is a closed two sided ideal then there exists a closed subset $S \subset X$ and for each $x \in S$ a closed two sided ideal $J_x \subset \boldsymbol{B}$ such that $J = \bigcap_{x \in S} J_x(x)$.*

PROPOSITION 5.2. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra, and $\mathfrak{A} = C(X;\boldsymbol{B})$. If $\alpha \in \mathrm{CF}\,(\mathfrak{A})$ and $x \in X$ then*

$$[\alpha(f)](x) = [\alpha(f(x))^{\sim}](x)$$

for all $f \in \mathfrak{A}$.

Proof. Let $f \in \mathfrak{A}$. Then $e_x(f-(f(x))^\sim)=0$. Thus by Lemma 3.2

$$e_x[\alpha(f-(f(x))^\sim)] = 0.$$

But this is by definition

$$[\alpha(f)](x)-[\alpha(f(x))^\sim](x) = 0,$$

and the result follows. □

REMARK. This result has been used at several key points in §III and points out the "local" nature of center-fixing automorphisms of $C(X; \boldsymbol{B})$.

LEMMA 5.3. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X; \boldsymbol{B})$. Let $J\subset \boldsymbol{B}$ be a closed two sided ideal and $x \in X$ a fixed point of X. Assume in addition that* CF $(\boldsymbol{B})\subseteq \tau_0(\boldsymbol{B})$. *Then for any $\alpha \in$* CF $(\mathfrak{A})$, $\alpha(J(x))=J(x)$.

Proof. Let $f \in J(x)$. By Proposition 5.2

$$(\alpha f)(x) = \alpha(f(x))^\sim(x) = \tilde{\alpha}_x(f(x)).$$

It follows directly from the definition of the function $^\sim$ given in §III that $\tilde{\alpha}_x$ is a center-fixing automorphism of $\boldsymbol{B}$. Since we have assumed that CF $(\boldsymbol{B})\subseteq \tau_0(\boldsymbol{B})$ it follows that $\tilde{\alpha}_x(f(x)) \in J$. Therefore $(\alpha f)(x) \in J$ and hence $f \in J(x)$. Thus $\alpha(J(x))\subseteq J(x)$. But α^{-1} is also in CF $(\mathfrak{A})$ and hence $\alpha^{-1}(J(x))\subseteq J(x)$. Applying α to both sides of this latter inclusion gives $J(x)\subseteq \alpha(J(x))$, and hence $\alpha(J(x))=J(x)$ as required. □

REMARKS. (1) The hypothesis CF $(\boldsymbol{B})\subseteq \tau_0(\boldsymbol{B})$ is often satisfied. For example if $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$, where $\boldsymbol{H}$ is a Hilbert space, then every automorphism of $\boldsymbol{B}$ is inner, and hence carefully ideal preserving. If $\boldsymbol{B}$ has only one closed two sided proper ideal, e.g. $\boldsymbol{B}=$the C^*-algebra of all compact operators on a Hilbert space $\boldsymbol{H}$ with identity adjoined, then every automorphism of $\boldsymbol{B}$ is carefully ideal preserving.

(2) Setting $X=$point we see that the assumption CF $(\boldsymbol{B})\subseteq \tau_0(\boldsymbol{B})$ is clearly necessary for the conclusion of Lemma 5.3 to hold.

The next proposition provides numerous additional examples of C^*-algebras such that CF $(\mathfrak{A})\subseteq \tau_0(\mathfrak{A})$.

PROPOSITION 5.4. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X; \boldsymbol{B})$. Assume in addition that* CF $(\boldsymbol{B})\subseteq \tau_0(\boldsymbol{B})$. *Then* CF $(\mathfrak{A})\subseteq \tau_0(\mathfrak{A})$.

Proof. It is immediate from Theorem H and Lemma 5.3. □

LEMMA 5.5. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra, $\mathfrak{A}=C(X; \boldsymbol{B})$, and $\alpha \in \tau(\mathfrak{A})$. Then for any $f \in \mathfrak{A}$*

$$(\alpha f)(x) = \alpha(f(x))^\sim(x),$$

for any $x \in X$.

Proof. Let $x \in X$ and set $J=\ker e_x$. Then J is a closed two sided ideal in $\mathfrak{A}$. Let $f \in \mathfrak{A}$. Then $f-(f(x))^\sim \in J$ and since α is ideal preserving $\alpha(f-(f(x))^\sim) \in J$, i.e. $\alpha(f)(x)=\alpha(f(x))^\sim(x)$, as required. □

LEMMA 5.6. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X;\boldsymbol{B})$. Let $\alpha \in \tau(\mathfrak{A})$. Then for each $x \in X$, $\tilde{\alpha}_x \in \tau(\boldsymbol{B})$.*

Proof. Let $x \in X$ and $J \subset \boldsymbol{B}$ a closed two sided ideal. Let $B \in J$. Then $\tilde{B} \in J(x)$. By definition of $\tilde{\alpha}_x$, $\tilde{\alpha}_x(B)=\alpha(\tilde{B})(x)$. Since $\alpha \in \tau(\mathfrak{A})$, $\alpha(J(x)) \subseteq J(x)$. Therefore $\alpha(\tilde{B}) \in J(x)$ and hence $\alpha(\tilde{B})(x) \in J$. Thus $\tilde{\alpha}_x(B) \in J$. Hence $\tilde{\alpha}_x(J) \subseteq J$ as required. □

By applying the identical argument to $\tilde{\alpha}_x^{-1}$ we obtain

LEMMA 5.7. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X;\boldsymbol{B})$. Let $\alpha \in \tau_0(\mathfrak{A})$. Then for each $x \in X$, $\tilde{\alpha}_x \in \tau_0(\boldsymbol{B})$.* □

PROPOSITION 5.8. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X;\boldsymbol{B})$. Assume in addition that $\tau(\boldsymbol{B}) \subseteq \mathrm{CF}(\boldsymbol{B})$. Then $\tau(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A})$.*

Proof. Let $\alpha \in \tau(\mathfrak{A})$ and $f \in \boldsymbol{Z}(\mathfrak{A})$. Then (since $\boldsymbol{Z}(\mathfrak{A})=C(X;\boldsymbol{Z}(\boldsymbol{B}))$) $f(x) \in \boldsymbol{Z}(\boldsymbol{B})$ for all $x \in X$. Thus for each $x \in X$ we have by Lemma 5.5

$$(\alpha f)(x) = \alpha(f(x))^{\sim}(x) = \alpha_x(f(x)).$$

By Lemma 5.6 $\alpha_x \in \tau(\boldsymbol{B})$ and hence by our hypothesis on $\boldsymbol{B}$, $\tilde{\alpha}_x \in \mathrm{CF}(\boldsymbol{B})$. Therefore $\tilde{\alpha}_x(f(x))=f(x)$. Thus $(\alpha f)(x)=f(x)$ and hence α is center-fixing. Thus $\tau(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A})$. □

REMARKS. (1) If $\boldsymbol{Z}(\boldsymbol{B})=\boldsymbol{C}\cdot I$ then $\mathrm{CF}(\boldsymbol{B})=\mathrm{Aut}(\boldsymbol{B})$ and thus clearly $\tau(\boldsymbol{B}) \subseteq \mathrm{CF}(\boldsymbol{B})$.

(2) $\boldsymbol{Z}(\boldsymbol{B})=\boldsymbol{C}\cdot I$ for $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$, $\boldsymbol{H}$ a Hilbert space, or $\boldsymbol{B}=\mathfrak{B}$, the C^*-algebra of all compact operators on a Hilbert space with identity adjoined. Thus our hypotheses are satisfied in these cases.

(3) Setting $X=$point shows that the hypotheses on $\boldsymbol{B}$ in Proposition 5.8 are necessary.

THEOREM 5.9. *Let X be a compact Hausdorff space, $\boldsymbol{B}$ a C^*-algebra and $\mathfrak{A}=C(X;\boldsymbol{B})$. Assume in addition that $\tau_0(\boldsymbol{B})=\mathrm{CF}(\boldsymbol{B})=\tau(\boldsymbol{B})$. Then*

$$\tau_0(\mathfrak{A}) = \mathrm{CF}(\mathfrak{A}) = \tau(\mathfrak{A}).$$

Proof. Since $\tau_0(\boldsymbol{B})=\mathrm{CF}(\boldsymbol{B})$ it follows from Proposition 5.4 that $\mathrm{CF}(\mathfrak{A}) \subseteq \tau_0(\mathfrak{A})$. Since $\tau(\boldsymbol{B})=\mathrm{CF}(\boldsymbol{B})$ it follows from Proposition 5.8 that $\tau(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A})$. Thus we have inclusions

$$\tau(\mathfrak{A}) \subseteq \mathrm{CF}(\mathfrak{A}) \subseteq \tau_0(\mathfrak{A}).$$

Since $\tau_0(\mathfrak{A}) \subseteq \tau(\mathfrak{A})$ by definition, the result follows. □

REMARKS. (1) The assumption that $\tau_0(\boldsymbol{B})=\mathrm{CF}(\boldsymbol{B})$ is clearly redundant, for $\tau(\boldsymbol{B})=\mathrm{CF}(\boldsymbol{B})$ then $\tau(\boldsymbol{B})$ is a group and as noted previously this implies $\tau(\boldsymbol{B})=\tau_0(\boldsymbol{B})$.

(2) Note that the hypotheses on $\boldsymbol{B}$ are satisfied when $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$, $\boldsymbol{H}$ a Hilbert space, or $\boldsymbol{B}=\mathfrak{B}$, the C^*-algebra of compact operators with identity adjoined.

COROLLARY 5.10. *Let $\mathfrak{A}$, $\boldsymbol{B}$ be as in Theorem 5.9. Then $\tau(\mathfrak{A})$ is a subgroup of* $\mathrm{Aut}(\mathfrak{A})$. □

COROLLARY 5.11. *Let X be a separable compact Hausdorff space, $\boldsymbol{B}=\mathscr{L}(\boldsymbol{H})$ and $\mathfrak{A}=C(X;\boldsymbol{B})$. Then*

$$\tau_0(\mathfrak{A}) = \mathrm{CF}(\mathfrak{A}) = \tau(\mathfrak{A}) = \pi(\mathfrak{A}).$$

Proof. Immediate from Corollary 4.2, Theorem 5.9 and Remark (1) after Lemma 5.3. □

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